

CONGRUENCES INVOLVING $\binom{2k}{k}^2 \binom{3k}{k} m^{-k}$

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ABSTRACT. Let $p > 3$ be a prime, and let m be an integer with $p \nmid m$. In the paper, based on the work of Brillhart and Morton, by using the work of Ishii and Deuring's theorem for elliptic curves with complex multiplication we solve some conjectures of Zhi-Wei Sun concerning $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} m^{-k} \pmod{p^2}$.

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1. Introduction.

For positive integers a, b and n , if $n = ax^2 + by^2$ for some integers x and y , we briefly say that $n = ax^2 + by^2$. Let $p > 3$ be a prime. In 2003, Rodriguez-Villegas[RV] posed some conjectures on supercongruences modulo p^2 . One of his conjectures is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

This conjecture has been solved by Mortenson[Mo] and Zhi-Wei Sun[Su2].

Let \mathbb{Z} be the set of integers, and for a prime p let \mathbb{Z}_p be the set of rational numbers whose denominator is coprime to p . Recently the author's brother Zhi-Wei Sun[Su1] posed many conjectures for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} m^{-k} \pmod{p^2}$, where $p > 3$ is a prime and $m \in \mathbb{Z}$ with $p \nmid m$. For example, he conjectured that (see [Su1, Conjecture A13])

$$(1.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p = 5x^2 + 3y^2 \equiv 2, 8 \pmod{15}. \end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$(1.2) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n,$$

where $[a]$ is the greatest integer not exceeding a . From (1.2) we see that

$$(1.3) \quad P_n(-x) = (-1)^n P_n(x).$$

Let $(\frac{a}{m})$ be the Jacobi symbol. For a prime $p > 3$, in [S2] the author showed that

$$(1.4) \quad P_{[\frac{p}{3}]}(t) \equiv \sum_{k=0}^{[p/3]} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54}\right)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54}\right)^k \pmod{p}.$$

We note that $p \mid \binom{2k}{k} \binom{3k}{k}$ for $\frac{p}{3} < k < p$. In the paper, using the work of Brillhart and Morton[BM] we prove that

$$(1.5) \quad P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \pmod{p}.$$

Based on (1.5) and the work of Ishii[I], we determine

$$P_{[\frac{p}{3}]}(t) \pmod{p} \quad \text{for} \quad t = \frac{5}{4}, \frac{5}{\sqrt{-2}}, \frac{\sqrt{-11}}{4}, \frac{1}{\sqrt{2}}, \sqrt{5}, \frac{9}{20}\sqrt{5}, \frac{\sqrt{17}}{4}, \frac{5}{32}\sqrt{41}, \frac{53}{500}\sqrt{89}.$$

For instance, if $p \equiv 1, 4 \pmod{5}$ is a prime, then

$$P_{[\frac{p}{3}]}(\sqrt{5}) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod{p} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Let $p > 3$ be a prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 108/m}$. In the paper we show that

$$(1.6) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv P_{[\frac{p}{3}]}(t)^2 \pmod{p}$$

and that

$$(1.7) \quad P_{[\frac{p}{3}]}(t) \equiv 0 \pmod{p} \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p^2}.$$

On the basis of (1.6) and (1.7), we prove some congruences for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} m^{-k}$ in the cases $m = 8, 64, 216, -27, -192, -8640, -12^3, -48^3, -300^3$. Thus we partially solve some conjectures posed by Zhi-Wei Sun in [Su1]. As two examples, for odd primes $p \neq 11$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv \begin{cases} x^2 \pmod{p} & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \end{cases}$$

for odd primes p with $(\frac{17}{p}) = 1$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-12)^{3k}} \equiv \begin{cases} x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

2. A general congruence modulo p^2 .

Lemma 2.1. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for $m = 0, 1$. Since both sides satisfy the same recurrence relation

$$81(m+1)(3m+2)(3m+4)S(m) - 3(2m+3)(9m^2+27m+22)S(m+1) + (m+2)^3 S(m+2) = 0,$$

we see that the lemma is true. The proof certificate for the left hand side is

$$-\frac{729k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{9k^2(3m-3k+1)(3m-3k+2)(9m^2-9mk+30m-14k+24)}{(m-k+1)^2(m-k+2)^2}.$$

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} x^k \sum_{r=0}^k \binom{k}{r} (-27x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p-2$ and $0 \leq k \leq p-1$. If $k > \frac{p}{2}$, then $p \mid \binom{2k}{k}$ and so $p^2 \mid \binom{2k}{k}^2$. If $k < \frac{p}{2}$, then $m-k \geq p-k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 2.1

we deduce that

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \\
& \equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k} \\
& = \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k} \\
& = \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k} x^{m-k} \\
& = \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{3r}{r} x^r \\
& = \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \left(\sum_{r=0}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r - \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \right) \\
& = \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \pmod{p^2}.
\end{aligned}$$

If $\frac{2p}{3} \leq k \leq p-1$, then $\binom{2k}{k} \binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p^2}$. If $0 \leq k \leq \frac{p}{3}$ and $p-k \leq r \leq p-1$, then $\frac{2p}{3} \leq r \leq p-1$ and so $\binom{2r}{r} \binom{3r}{r} = \frac{(3r)!}{r!^3} \equiv 0 \pmod{p^2}$. If $\frac{p}{3} < k < \frac{2p}{3}$ and $p-k \leq r \leq p-1$, then $r \geq p-k > \frac{p}{3}$, $\binom{2k}{k} \binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p}$ and $\binom{2r}{r} \binom{3r}{r} = \frac{(3r)!}{r!^3} \equiv 0 \pmod{p}$. Hence, for $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$ we have $p^2 \mid \binom{2k}{k} \binom{3k}{k} \binom{2r}{r} \binom{3r}{r}$ and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \equiv 0 \pmod{p^2}.$$

Therefore the result follows.

Corollary 2.1. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1 - \sqrt{1 - 108/m}}{54} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 108/m}}{54}$ in Theorem 2.1 we deduce the result.

3. Congruences for $P_{[p/3]}(t) \pmod{p}$.

Let $W_n(x)$ be the Deuring polynomial given by

$$(3.1) \quad W_n(x) = \sum_{k=0}^n \binom{n}{k}^2 x^k.$$

It is known that ([G,(3.134)],[BM])

$$(3.2) \quad W_n(x) = (1-x)^n P_n\left(\frac{1+x}{1-x}\right).$$

Let $p > 3$ be a prime, $m, n \in \mathbb{Z}_p$ and $4m^3 + 27n^2 \not\equiv 0 \pmod{p}$. From [Mor, Theorem 3.3] we have

$$(3.3) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right) \equiv -(-48m)^{\frac{1-(\frac{p}{3})}{2}} (864n)^{\frac{1-(\frac{-1}{p})}{2}} (-16(4m^3 + 27n^2))^{\lfloor \frac{p}{12} \rfloor} J_p\left(\frac{2^8 \cdot 3^3 m^3}{4m^3 + 27n^2}\right) \pmod{p},$$

where $J_p(t)$ is a certain Jacobi polynomial given by

$$(3.4) \quad J_p(t) = 1728^{\lfloor \frac{p}{12} \rfloor} P_{\lfloor \frac{p}{12} \rfloor}^{(-\frac{1}{3}(\frac{p}{3}), -\frac{1}{2}(\frac{-1}{p}))} \left(1 - \frac{t}{864}\right)$$

and

$$P_k^{(\alpha, \beta)}(x) = \frac{1}{2^k} \sum_{r=0}^k \binom{k+\alpha}{r} \binom{k+\beta}{k-r} (x-1)^{k-r} (x+1)^r.$$

Theorem 3.1. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$P_{\lfloor \frac{p}{3} \rfloor}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \pmod{p}.$$

Proof. It is well known that $P_n(1) = 1$. Since $P_{\lfloor \frac{p}{3} \rfloor}(1) = 1$ and

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 3x - 2}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{(x+1)^2(x-2)}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x-2}{p} \right) - \left(\frac{-1-2}{p} \right) = -\left(\frac{p}{3}\right),$$

we see that the result is true for $t \equiv 1 \pmod{p}$. Since $P_{\lfloor \frac{p}{3} \rfloor}(-1) = (-1)^{\lfloor \frac{p}{3} \rfloor} P_{\lfloor \frac{p}{3} \rfloor}(1) = \left(\frac{p}{3}\right)$ and

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 27x + 54}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{(-3x)^3 - 27(-3x) + 54}{p} \right) = \left(\frac{-3}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3x - 2}{p} \right) = -1,$$

we see that the result is also true for $t \equiv -1 \pmod{p}$.

Now we assume $t \not\equiv \pm 1 \pmod{p}$. Set $W_n(x) = \sum_{k=0}^n \binom{n}{k}^2 x^k$. From [BM, Theorem 6] we know that

$$W_{\lfloor \frac{p}{3} \rfloor} \left(1 - \frac{x}{27}\right) \equiv u_p(x) (x-27)^{\lfloor \frac{p}{12} \rfloor} J_p \left(\frac{x(x-24)^3}{x-27} \right) \pmod{p},$$

where $J_p(x)$ is a certain Jacobi polynomial given by (3.4) and

$$u_p(x) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12}, \\ -3(x-24) & \text{if } p \equiv 5 \pmod{12}, \\ x^2 - 36x + 216 & \text{if } p \equiv 7 \pmod{12}, \\ -3(x-24)(x^2 - 36x + 216) & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Set $x = 54/(t+1)$. We then have

$$(3.5) \quad W_{[\frac{p}{3}]}((t-1)/(t+1)) \equiv \begin{cases} \left(\frac{27(1-t)}{1+t}\right)^{[\frac{p}{12}]} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \pmod{p} & \text{if } p \equiv 1 \pmod{12}, \\ \frac{18(4t-5)}{t+1} \left(\frac{27(1-t)}{1+t}\right)^{[\frac{p}{12}]} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \pmod{p} & \text{if } p \equiv 5 \pmod{12}, \\ \frac{108(2t^2-14t+11)}{(t+1)^2} \left(\frac{27(1-t)}{1+t}\right)^{[\frac{p}{12}]} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \pmod{p} & \text{if } p \equiv 7 \pmod{12}, \\ \frac{1944(4t-5)(2t^2-14t+11)}{(t+1)^3} \left(\frac{27(1-t)}{1+t}\right)^{[\frac{p}{12}]} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \pmod{p} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

By (3.2) we have

$$(3.6) \quad W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1} \right) = \left(1 - \frac{t-1}{t+1} \right)^{[\frac{p}{3}]} P_{[\frac{p}{3}]} \left(\frac{1 + (t-1)/(t+1)}{1 - (t-1)/(t+1)} \right) = \left(\frac{2}{t+1} \right)^{[\frac{p}{3}]} P_{[\frac{p}{3}]}(t).$$

If $p \equiv 2 \pmod{3}$ and $t \equiv \frac{5}{4} \pmod{p}$, from the above we get

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) = \left(\frac{\frac{5}{4} + 1}{2} \right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{\frac{5}{4} - 1}{\frac{5}{4} + 1} \right) \equiv 0 \pmod{p}.$$

On the other hand,

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x^3 - 27/4}{p} \right) = \sum_{y=0}^{p-1} \left(\frac{y - 27/4}{p} \right) = 0.$$

Thus the result is true when $p \equiv 2 \pmod{3}$ and $t \equiv \frac{5}{4} \pmod{p}$. If $p \equiv 3 \pmod{4}$ and $2t^2 - 14t + 11 \equiv 0 \pmod{p}$, from (3.5) and (3.6) we deduce that

$$P_{[\frac{p}{3}]}(t) = \left(\frac{t+1}{2} \right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1} \right) \equiv 0 \pmod{p}.$$

As

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{(-x)^3 + 3(4t-5)(-x)}{p} \right) = - \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x}{p} \right),$$

we see that

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x}{p} \right) = 0.$$

Thus the result is true when $p \equiv 3 \pmod{4}$ and $2t^2 - 14t + 11 \equiv 0 \pmod{p}$. Set $m = 3(4t - 5)$ and $n = 2(2t^2 - 14t + 11)$. Then

$$4m^3 + 27n^2 = -432(1-t)(1+t)^3 \quad \text{and so} \quad \frac{2^8 \cdot 3^3 m^3}{4m^3 + 27n^2} = \frac{432(5-4t)^3}{(1-t)(1+t)^3}.$$

By the above we need only to assume $m \not\equiv 0 \pmod{p}$ for $p \equiv 2 \pmod{3}$ and $n \not\equiv 0 \pmod{p}$ for $p \equiv 3 \pmod{4}$. From (3.3) we see that

$$\begin{aligned} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) &= J_p\left(\frac{2^8 \cdot 3^3 m^3}{4m^3 + 27n^2}\right) \\ &\equiv -(-48m)^{\frac{(\frac{p}{3})-1}{2}} (864n)^{\frac{(\frac{-1}{p})-1}{2}} (-16(4m^3 + 27n^2))^{-[\frac{p}{12}]} \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \pmod{p}. \end{aligned}$$

If $p \equiv 1 \pmod{12}$, from all the above we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]}(t) &= \left(\frac{t+1}{2}\right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1}\right) \equiv \left(\frac{t+1}{2}\right)^{\frac{p-1}{3}} \left(\frac{27(1-t)}{1+t}\right)^{\frac{p-1}{12}} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \\ &\equiv -2^{-\frac{p-1}{3}} (3(t+1))^{\frac{p-1}{4}} (1-t)^{\frac{p-1}{12}} (16 \cdot 432(1-t)(1+t)^3)^{-\frac{p-1}{12}} \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\ &\equiv -\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p}\right) \pmod{p}. \end{aligned}$$

If $p \equiv 5 \pmod{12}$, from all the above we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]}(t) &= \left(\frac{t+1}{2}\right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1}\right) \equiv \left(\frac{t+1}{2}\right)^{\frac{p-2}{3}} \frac{18(4t-5)}{t+1} \left(\frac{27(1-t)}{1+t}\right)^{\frac{p-5}{12}} J_p\left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \\ &\equiv 2^{-\frac{p-5}{3}} 3^{\frac{p+3}{4}} (4t-5)(1+t)^{\frac{p-5}{4}} (1-t)^{\frac{p-5}{12}} (144(4t-5))^{-1} \\ &\quad \times (16 \cdot 432(1-t)(1+t)^3)^{-\frac{p-5}{12}} \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\ &\equiv \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p}\right) \pmod{p}. \end{aligned}$$

If $p \equiv 7 \pmod{12}$, from all the above we deduce that

$$\begin{aligned}
P_{[\frac{p}{3}]}(t) &= \left(\frac{t+1}{2}\right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1}\right) \\
&\equiv \left(\frac{t+1}{2}\right)^{\frac{p-1}{3}} \frac{108(2t^2 - 14t + 11)}{(t+1)^2} \left(\frac{27(1-t)}{1+t}\right)^{\frac{p-7}{12}} J_p \left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \\
&\equiv -2^{-\frac{p-7}{3}} 3^{\frac{p+5}{4}} (2t^2 - 14t + 11)(1+t)^{\frac{p-7}{4}} (1-t)^{\frac{p-7}{12}} (1728(2t^2 - 14t + 11))^{-1} \\
&\quad \times (16 \cdot 432(1-t)(1+t)^3)^{-\frac{p-7}{12}} \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\
&\equiv -\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p}\right) \pmod{p}.
\end{aligned}$$

If $p \equiv 11 \pmod{12}$, from all the above we deduce that

$$\begin{aligned}
P_{[\frac{p}{3}]}(t) &= \left(\frac{t+1}{2}\right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{t-1}{t+1}\right) \\
&\equiv \left(\frac{t+1}{2}\right)^{\frac{p-2}{3}} \frac{1944(4t-5)(2t^2 - 14t + 11)}{(t+1)^3} \left(\frac{27(1-t)}{1+t}\right)^{\frac{p-11}{12}} J_p \left(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\right) \\
&\equiv 2^{-\frac{p-11}{3}} 3^{\frac{p-11}{4}+5} (4t-5)(2t^2 - 14t + 11)(1+t)^{\frac{p-11}{4}} (1-t)^{\frac{p-11}{12}} (48m)^{-1} (864n)^{-1} \\
&\quad \times (16 \cdot 432(1-t)(1+t)^3)^{-\frac{p-11}{12}} \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\
&\equiv \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p}\right) \pmod{p}.
\end{aligned}$$

This proves the theorem.

Corollary 3.1. *Let $p > 3$ be a prime and let t be a variable. Then*

$$\begin{aligned}
&\sum_{k=0}^{[p/3]} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54}\right)^k \\
&\equiv P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} (x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11))^{\frac{p-1}{2}} \pmod{p}.
\end{aligned}$$

Proof. From [S2, Lemma 2.3] we have $P_{[\frac{p}{3}]}(t) \equiv \sum_{k=0}^{[p/3]} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54}\right)^k \pmod{p}$. By Theorem 3.1 and Euler's criterion, the result is true for $t = 0, 1, \dots, p-1$. Since both sides are polynomials of t with degree at most $p-1$. Using Lagrange's theorem we obtain the result.

Corollary 3.2. *Let $p \geq 17$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \\ &= \left(\frac{p}{3} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(4t+5)x + 2(2t^2 + 14t + 11)}{p} \right). \end{aligned}$$

Proof. Since $P_{[\frac{p}{3}]}(-t) = (-1)^{[\frac{p}{3}]} P_{[\frac{p}{3}]}(t) = \left(\frac{p}{3} \right) P_{[\frac{p}{3}]}(t)$, by Theorem 3.1 we have

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \\ & \equiv \left(\frac{p}{3} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(4t+5)x + 2(2t^2 + 14t + 11)}{p} \right) \pmod{p}. \end{aligned}$$

By Weil's estimate ([BEW, p.183]) we have

$$\begin{aligned} & \left| \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \right| \leq 2\sqrt{p}, \\ & \left| \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(4t+5)x + 2(2t^2 + 14t + 11)}{p} \right) \right| \leq 2\sqrt{p}. \end{aligned}$$

Since $4\sqrt{p} < p$ for $p \geq 17$, from the above we deduce the result.

Corollary 3.3. *Let $p > 3$ be a prime. Then*

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 120x + 506}{p} \right) = \begin{cases} \left(\frac{2}{p} \right) L & \text{if } 3 \mid p-1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. It is easy to check the result for $p = 5, 7, 11, 13$. Now we assume $p \geq 17$. Taking $t = \frac{5}{4}$ in Corollary 3.2 we find that

$$\begin{aligned} \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{27}{4}}{p} \right) &= \left(\frac{p}{3} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 30x + \frac{253}{4}}{p} \right) = \left(\frac{p}{3} \right) \sum_{x=0}^{p-1} \left(\frac{\left(\frac{x}{2} \right)^3 - 30 \cdot \frac{x}{2} + \frac{253}{4}}{p} \right) \\ &= \left(\frac{p}{3} \right) \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 120x + 506}{p} \right). \end{aligned}$$

For $p \equiv 2 \pmod{3}$ it is clear that

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{27}{4}}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x - \frac{27}{4}}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x}{p} \right) = 0.$$

Thus the result is true when $p \equiv 2 \pmod{3}$.

Now assume $p \equiv 1 \pmod{3}$, $p = A^2 + 3B^2$, $4p = L^2 + 27M^2$ and $A \equiv L \equiv 1 \pmod{3}$. It is known that $2^{\frac{p-1}{3}} \equiv 1 \pmod{p}$ if and only if $3 \mid B$. When $3 \nmid B$ we choose the sign of B so that $B \equiv 1 \pmod{3}$. By [S1, (2.12)] we have $2^{(p-1)/3} \equiv \frac{1}{2}(-1 - \frac{A}{B}) \pmod{p}$. From [S1, (2.9)-(2.11)] we deduce that

$$\begin{aligned} \sum_{x=0}^{p-1} \left(\frac{x^3 - 27/4}{p} \right) &= 1 + \sum_{x=1}^{p-1} \left(\frac{x^3 - 27/4}{p} \right) \\ &= \begin{cases} -2A = L \pmod{p} & \text{if } 2^{\frac{p-1}{3}} \equiv 1 \pmod{p}, \\ A + 3B = L \pmod{p} & \text{if } 2^{\frac{p-1}{3}} \not\equiv 1 \pmod{p} \text{ and } B \equiv 1 \pmod{3}. \end{cases} \end{aligned}$$

Thus

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 120x + 506}{p} \right) = \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 27/4}{p} \right) = \left(\frac{2}{p} \right) L.$$

This completes the proof.

Theorem 3.2. *Let $p > 3$ be a prime. Then*

(i) *If $p \equiv 2 \pmod{3}$, then*

$$\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{(-216)^k} \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{24^k} \equiv P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv 0 \pmod{p}.$$

(ii) *If $p \equiv 1 \pmod{3}$ and so $4p = L^2 + 27M^2$ with $L, M \in \mathbb{Z}$ and $L \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{(-216)^k} \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{24^k} \equiv P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv -L \equiv \left(\frac{-2}{p} \right) \left(\frac{\frac{2(p-1)}{3}}{[\frac{p}{12}]} \right) \pmod{p}.$$

Proof. Putting $t = \pm \frac{5}{4}$ in Corollary 3.1 we get

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{(-216)^k} \pmod{p} \quad \text{and} \quad P_{[\frac{p}{3}]} \left(-\frac{5}{4} \right) \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{24^k} \pmod{p}.$$

This together with (1.3) yields

$$\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{(-216)^k} \equiv \left(\frac{p}{3} \right) \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k} \binom{3k}{k}}{24^k} \pmod{p}.$$

If $p \equiv 2 \pmod{3}$, by Theorem 3.1 we have

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{27}{4}}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x - \frac{27}{4}}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x}{p} \right) = 0 \pmod{p}.$$

Thus (i) is true.

Now assume $p \equiv 1 \pmod{3}$, $4p = L^2 + 27M^2$ and $L \equiv 1 \pmod{3}$. By Theorem 3.1 and the proof of Corollary 3.3 we have

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv - \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{27}{4}}{p} \right) = -L \pmod{p}.$$

On the other hand, by the proof of Theorem 3.1,

$$\begin{aligned} P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) &= \left(\frac{\frac{5}{4} + 1}{2} \right)^{[\frac{p}{3}]} W_{[\frac{p}{3}]} \left(\frac{\frac{5}{4} - 1}{\frac{5}{4} + 1} \right) \\ &\equiv \begin{cases} \left(\frac{9}{8} \right)^{\frac{p-1}{3}} \left(\frac{27(1-\frac{5}{4})}{1+\frac{5}{4}} \right)^{\frac{p-1}{12}} J_p(0) \equiv (-1)^{\frac{p-1}{12}} 3^{-\frac{p-1}{4}} J_p(0) \pmod{p} & \text{if } p \equiv 1 \pmod{12}, \\ \left(\frac{9}{8} \right)^{\frac{p-1}{3}} \frac{108(2(\frac{5}{4})^2 - 14 \cdot \frac{5}{4} + 11)}{(\frac{5}{4} + 1)^2} \left(\frac{27(1-\frac{5}{4})}{1+\frac{5}{4}} \right)^{\frac{p-7}{12}} J_p(0) \\ \equiv -8(-1)^{\frac{p-7}{12}} 3^{-\frac{p-7}{4}} J_p(0) \pmod{p} & \text{if } p \equiv 7 \pmod{12}. \end{cases} \end{aligned}$$

By the definition of $J_p(x)$, we have

$$\begin{aligned} J_p(0) &= 1728^{[\frac{p}{12}]} \cdot 2^{-[\frac{p}{12}]} \sum_{r=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}] - \frac{1}{3}(\frac{p}{3})}{r} \binom{[\frac{p}{12}] - \frac{1}{2}(\frac{-1}{p})}{[\frac{p}{12}] - r} 0^{[\frac{p}{12}] - r} 2^r \\ &= 1728^{[\frac{p}{12}]} \binom{[\frac{p}{12}] - \frac{1}{3}(\frac{p}{3})}{[\frac{p}{12}]} = (-1728)^{[\frac{p}{12}]} \binom{\frac{1}{3}(\frac{p}{3}) - 1}{[\frac{p}{12}]} \pmod{p}. \end{aligned}$$

Hence

$$J_p(0) \equiv (-1728)^{[\frac{p}{12}]} \binom{\frac{2(p-1)}{3}}{[\frac{p}{12}]} \pmod{p}$$

and therefore

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv \begin{cases} (-1)^{\frac{p-1}{12}} 3^{-\frac{p-1}{4}} (-1728)^{\frac{p-1}{12}} \binom{\frac{2(p-1)}{3}}{\frac{p-1}{12}} \equiv \left(\frac{2}{p} \right) \binom{\frac{2(p-1)}{3}}{\frac{p-1}{12}} \pmod{p} & \text{if } 12 \mid p-1, \\ -8(-1)^{\frac{p-7}{12}} 3^{-\frac{p-7}{4}} (-1728)^{\frac{p-7}{12}} \binom{\frac{2(p-1)}{3}}{\frac{p-7}{12}} \equiv -\left(\frac{2}{p} \right) \binom{\frac{2(p-1)}{3}}{\frac{p-7}{12}} \pmod{p} & \text{if } 12 \mid p-7. \end{cases}$$

Now putting all the above together we obtain the result.

Remark 3.1 For any prime $p > 3$, Zhi-Wei Sun conjectured that ([Su1, Conjecture A46])

$$\sum_{k=0}^{p-1} \frac{(3k)!}{24^k \cdot k!^3} \equiv \left(\frac{p}{3} \right) \sum_{k=0}^{p-1} \frac{(3k)!}{(-216)^k \cdot k!^3} \equiv \begin{cases} \binom{(2(p-1)/3)}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

4. Congruences for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k}$.

Let $p > 3$ be a prime and $m \in \mathbb{Z}$ with $p \nmid m$. In the section we partially solve Z.W. Sun's conjectures on $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} m^{-k} \pmod{p^2}$.

Theorem 4.1. *Let $p > 3$ be a prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 108/m}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv P_{[\frac{p}{3}]}(t)^2 \equiv \left(\sum_{x=0}^{p-1} (x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11))^{\frac{p-1}{2}} \right)^2 \pmod{p}.$$

Moreover, if $P_{[\frac{p}{3}]}(t) \equiv 0 \pmod{p}$ or $\sum_{x=0}^{p-1} (x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11))^{\frac{p-1}{2}} \equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $\frac{1-t}{54}(1 - 27 \cdot \frac{1-t}{54}) = \frac{1}{m}$, by Theorem 2.1 we have

$$(4.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54} \right)^k \right)^2 \pmod{p^2}.$$

Observe that $p \mid \binom{2k}{k} \binom{3k}{k}$ for $[\frac{p}{3}] < k < p$. From the above and Corollary 3.1 we see that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1-t}{54} \right)^k \\ & \equiv P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3} \right) \sum_{x=0}^{p-1} (x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11))^{\frac{p-1}{2}} \pmod{p}. \end{aligned}$$

This together with (4.1) yields the result.

Theorem 4.2 ([Su1, Conjecture A8]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-192)^k} \equiv \begin{cases} L^2 \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = L^2 + 27M^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. Putting $m = -192$ and $t = \frac{5}{4}$ in Theorem 4.1 and then applying Theorem 3.2 we obtain the result.

Lemma 4.1. *Let p be an odd prime and let a, m, n be p -adic integers. Then*

$$\sum_{x=0}^{p-1} (x^3 + a^2mx + a^3n)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \pmod{p}.$$

Moreover, if a, m, n are congruent to some integers, then

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + a^2mx + a^3n}{p} \right) = \left(\frac{a}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Proof. For any positive integer k it is well known that (see [IR])

$$\sum_{x=0}^{p-1} x^k \equiv \begin{cases} p-1 \pmod{p} & \text{if } p-1 \mid k, \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

Since

$$\begin{aligned} & \sum_{x=0}^{p-1} (x^3 + a^2mx + a^3n)^{\frac{p-1}{2}} \\ &= \sum_{x=0}^{p-1} \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} (x^3 + a^2mx)^k (a^3n)^{\frac{p-1}{2}-k} \\ &= \sum_{x=0}^{p-1} \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \sum_{r=0}^k \binom{k}{r} x^{3r} (a^2mx)^{k-r} (a^3n)^{\frac{p-1}{2}-k} \\ &= \sum_{r=0}^{(p-1)/2} \sum_{k=r}^{(p-1)/2} \binom{(p-1)/2}{k} \binom{k}{r} (a^2m)^{k-r} (a^3n)^{\frac{p-1}{2}-k} \sum_{x=0}^{p-1} x^{k+2r} \\ &\equiv (p-1) \sum_{r=0}^{(p-1)/2} \binom{(p-1)/2}{p-1-2r} \binom{p-1-2r}{r} (a^2m)^{p-1-3r} (a^3n)^{2r-\frac{p-1}{2}} \\ &= a^{\frac{p-1}{2}} (p-1) \sum_{\frac{p-1}{4} \leq r \leq \frac{p-1}{3}} \binom{(p-1)/2}{p-1-2r} \binom{p-1-2r}{r} m^{p-1-3r} n^{2r-\frac{p-1}{2}} \pmod{p}, \end{aligned}$$

we see that the congruence in Lemma 4.1 is true.

Now suppose that a, m, n are congruent to some integers. If $a \equiv 0 \pmod{p}$, clearly

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + a^2mx + a^3n}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x^3}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x}{p} \right) = 0 = \left(\frac{a}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

If $a \not\equiv 0 \pmod{p}$, then clearly

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + a^2mx + a^3n}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{(ax)^3 + a^2m(ax) + a^3n}{p} \right) = \left(\frac{a}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Thus the lemma is proved.

Lemma 4.2. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 - 30x - 56}{p} \right) \\ &= \begin{cases} (-1)^{\lfloor \frac{p}{8} \rfloor + 1} \left(\frac{3}{p} \right) 2c & \text{if } p \equiv 1, 3 \pmod{8}, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ 0 & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Proof. From [BE, Theorems 5.12 and 5.17] we know that

$$\sum_{k=0}^{p-1} \left(\frac{x^3 - 4x^2 + 2x}{p} \right) = \begin{cases} (-1)^{\lfloor \frac{p}{8} \rfloor + 1} 2c & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ with } 4 \mid c - 1, \\ 0 & \text{otherwise.} \end{cases}$$

As $27(x^3 - 4x^2 + 2x) = (3x - 4)^3 - 30(3x - 4) - 56$, we see that

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 - 4x^2 + 2x}{p} \right) \\ &= \left(\frac{3}{p} \right) \sum_{x=0}^{p-1} \left(\frac{(3x - 4)^3 - 30(3x - 4) - 56}{p} \right) = \left(\frac{3}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 30x - 56}{p} \right). \end{aligned}$$

Thus the result follows.

Lemma 4.3. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} (n^3 - (15 + 30\sqrt{-2})n - 28 + 70\sqrt{-2})^{\frac{p-1}{2}} \\ & \equiv \begin{cases} \left(\frac{2+\sqrt{-2}}{p} \right) (-1)^{\lfloor \frac{p}{8} \rfloor + 1} \left(\frac{3}{p} \right) 2c \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ and } 4 \mid c - 1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Proof. It is easily seen that

$$-15(1 + 2\sqrt{-2}) = -30 \left(\frac{1 - \sqrt{-2}}{\sqrt{-2}} \right)^2 \quad \text{and} \quad -28 + 70\sqrt{-2} = -56 \left(\frac{1 - \sqrt{-2}}{\sqrt{-2}} \right)^3.$$

Thus, by Lemmas 4.1 and 4.2 we have

$$\begin{aligned} & \sum_{n=0}^{p-1} (n^3 - (15 + 30\sqrt{-2})n - 28 + 70\sqrt{-2})^{\frac{p-1}{2}} \\ & \equiv \left(\frac{1 - \sqrt{-2}}{\sqrt{-2}} \right)^{\frac{p-1}{2}} \sum_{n=0}^{p-1} (n^3 - 30n - 56)^{\frac{p-1}{2}} \\ & \equiv \left(-\frac{2 + \sqrt{-2}}{2} \right)^{\frac{p-1}{2}} \sum_{n=0}^{p-1} \left(\frac{n^3 - 30n - 56}{p} \right) \\ & \equiv \begin{cases} \left(\frac{2+\sqrt{-2}}{p} \right) (-1)^{\lfloor \frac{p}{8} \rfloor + 1} \left(\frac{3}{p} \right) 2c \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ and } 4 \mid c - 1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

This proves the lemma.

Theorem 4.3. *Let p be an odd prime. Then*

$$P_{[\frac{p}{3}]}(5/\sqrt{-2}) \equiv \begin{cases} (-1)^{[\frac{p}{8}]}(\frac{-2-\sqrt{-2}}{p})2c \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ and } 4 \mid c-1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \equiv \begin{cases} 4c^2 \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From Corollary 3.1 and Lemma 4.3 we deduce that

$$P_{[\frac{p}{3}]}(\frac{5}{\sqrt{-2}}) \equiv -\left(\frac{p}{3}\right) \sum_{n=0}^{p-1} (n^3 + 3(-10\sqrt{-2} - 5)n - 28 + 70\sqrt{-2})^{\frac{p-1}{2}} \\ \equiv \begin{cases} \left(\frac{p}{3}\right)(\frac{2+\sqrt{-2}}{p})(-1)^{[\frac{p}{8}]}(\frac{3}{p})2c = (\frac{-2-\sqrt{-2}}{p})(-1)^{[\frac{p}{8}]}2c \pmod{p} \\ \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ and } 4 \mid c-1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Now taking $m = 8$ and $t = 5/\sqrt{-2}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Remark 4.1 Let p be an odd prime. Zhi-Wei Sun conjectured that ([Su1, Conjecture A5])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \equiv \begin{cases} 4c^2 - 2p \pmod{p^2} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Corollary 4.1. *Let p be a prime such that $p \equiv 1, 3 \pmod{8}$ and $p = c^2 + 2d^2$ with $c \equiv 1 \pmod{4}$. Then*

$$P_{[\frac{p}{3}]}(\frac{5d}{c}) \equiv \begin{cases} (\frac{c-d}{3})2c \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{d-1}{2}}(\frac{c-d}{3})2c \pmod{p} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Proof. It is clear that $(\frac{d}{p}) = (\frac{p}{d}) = (\frac{c^2}{d}) = 1$. Thus

$$\begin{aligned} \left(\frac{2+c/d}{p}\right) &= \left(\frac{d}{p}\right)\left(\frac{c+2d}{p}\right) = \left(\frac{c+2d}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{2d+c-1}{2}} \left(\frac{p}{c+2d}\right) \\ &= (-1)^{\frac{p-1}{2}d} \left(\frac{(c+2d)(c-2d) + 6d^2}{c+2d}\right) = (-1)^{\frac{p-1}{2}d} \left(\frac{6}{c+2d}\right) \\ &= (-1)^{\frac{p-1}{2}d + \frac{(c+2d)^2-1}{8}} \left(\frac{3}{c+2d}\right) = (-1)^{\frac{p-1+4d+2d^2}{8}} \cdot (-1)^{\frac{p-1}{2}d + \frac{c+2d-1}{2}} \left(\frac{c+2d}{3}\right) \\ &= (-1)^{\frac{p-1+4d+2d^2}{8}} \left(\frac{c-d}{3}\right) = \begin{cases} (-1)^{\frac{p-1}{8}}(\frac{c-d}{3}) & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{p-3}{8} + \frac{d+1}{2}}(\frac{c-d}{3}) & \text{if } p \equiv 3 \pmod{8}. \end{cases} \end{aligned}$$

By Theorem 4.3 we have

$$P_{[\frac{p}{3}]}(\frac{5}{c/d}) \equiv (-1)^{[\frac{p}{8}]} \left(\frac{-1}{p}\right) \left(\frac{2+c/d}{p}\right) 2c \pmod{p}.$$

Now combining all the above we obtain the result.

Lemma 4.4. *Let p be an odd prime and $p \neq 11$. Then*

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 24 \cdot 11x + 14 \cdot 11^2}{p} \right) = \begin{cases} \left(\frac{u}{11} \right) u & \text{if } \left(\frac{p}{11} \right) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 & \text{if } \left(\frac{p}{11} \right) = -1. \end{cases}$$

Proof. It is known that (see [PR] and [JM])

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 96 \cdot 11x + 112 \cdot 11^2}{p} \right) = \begin{cases} \left(\frac{2}{p} \right) \left(\frac{u}{11} \right) u & \text{if } \left(\frac{p}{11} \right) = 1 \text{ and } 4p = u^2 + 11v^2, \\ 0 & \text{if } \left(\frac{p}{11} \right) = -1. \end{cases}$$

Since

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 - 96 \cdot 11x + 112 \cdot 11^2}{p} \right) \\ &= \sum_{x=0}^{p-1} \left(\frac{(2x)^3 - 96 \cdot 11 \cdot 2x + 112 \cdot 11^2}{p} \right) = \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 24 \cdot 11x + 14 \cdot 11^2}{p} \right), \end{aligned}$$

we deduce the result.

Lemma 4.5. *Let $p \neq 11$ be an odd prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} (n^3 + 12(-5 + \sqrt{-11})n + 14(11 - 4\sqrt{-11}))^{\frac{p-1}{2}} \\ & \equiv \begin{cases} \left(\frac{-22+2\sqrt{-11}}{p} \right) \left(\frac{u}{11} \right) u \pmod{p} & \text{if } \left(\frac{p}{11} \right) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{11} \right) = -1. \end{cases} \end{aligned}$$

Proof. It is easily seen that

$$12(-5 + \sqrt{-11}) = -24 \cdot 11 \left(\frac{\sqrt{-11} + 1}{2\sqrt{-11}} \right)^2 \text{ and } 14(11 - 4\sqrt{-11}) = 14 \cdot 11^2 \left(\frac{\sqrt{-11} + 1}{2\sqrt{-11}} \right)^3.$$

Thus, by Lemma 4.1 we have

$$\begin{aligned} & \sum_{n=0}^{p-1} (n^3 + 12(-5 + \sqrt{-11})n + 14(11 - 4\sqrt{-11}))^{\frac{p-1}{2}} \\ & \equiv \left(\frac{\sqrt{-11} + 1}{2\sqrt{-11}} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} (x^3 - 24 \cdot 11x + 14 \cdot 11^2)^{\frac{p-1}{2}} \\ & \equiv \left(\frac{-22 + 2\sqrt{-11}}{-11 \cdot 4} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 24 \cdot 11x + 14 \cdot 11^2}{p} \right) \pmod{p}. \end{aligned}$$

Now applying Lemma 4.4 we deduce the result.

Theorem 4.4. *Let $p \neq 11$ be an odd prime. Then*

$$P_{[\frac{p}{3}]} \left(\frac{\sqrt{-11}}{4} \right) \equiv \begin{cases} -(\frac{p}{3})(\frac{-11+\sqrt{-11}}{p})(\frac{u}{11})u \pmod{p} & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{11}) = -1 \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv \begin{cases} u^2 \pmod{p} & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

Proof. From Corollary 3.1 and Lemma 4.5 we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]} \left(\frac{\sqrt{-11}}{4} \right) &\equiv -\left(\frac{p}{3}\right) \sum_{n=0}^{p-1} \left(n^3 + 3(\sqrt{-11} - 5)n + \frac{-11}{4} + 22 - 7\sqrt{-11} \right)^{\frac{p-1}{2}} \\ &\equiv -\left(\frac{p}{3}\right) \sum_{n=0}^{p-1} \left(\left(\frac{n}{2}\right)^3 + 3(\sqrt{-11} - 5)\frac{n}{2} + \frac{77 - 28\sqrt{-11}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv -\left(\frac{p}{3}\right) \left(\frac{2}{p}\right) \sum_{n=0}^{p-1} \left(n^3 + 12(-5 + \sqrt{-11})n + 14(11 - 4\sqrt{-11}) \right)^{\frac{p-1}{2}} \\ &\equiv \begin{cases} -(\frac{p}{3})(\frac{-11+\sqrt{-11}}{p})(\frac{u}{11})u \pmod{p} & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{11}) = -1. \end{cases} \end{aligned}$$

Now taking $m = 64$ and $t = \frac{\sqrt{-11}}{4}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Remark 4.2 Let p be an odd prime such that $p \neq 11$. Zhi-Wei Sun conjectured that ([Sul, Conjecture A4])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv \begin{cases} u^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = u^2 + 11v^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

Corollary 4.2. *Let p be an odd prime such that $(\frac{p}{11}) = 1$ and so $4p = u^2 + 11v^2$ with $u, v \in \mathbb{Z}$. Let $v = 2^\alpha v_0$ and $u - 11v = 2^\beta w$ with $2 \nmid v_0 w$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{u}{4v} \right) \equiv (-1)^{\frac{p-1}{2} \cdot \frac{\alpha-\beta}{2}} \left(\frac{2}{p} \right)^{\alpha+\beta} \left(\frac{v-u}{3} \right) u \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned}
\left(\frac{-11+u/v}{p}\right) &= \left(\frac{v(u-11v)}{p}\right) = \left(\frac{2}{p}\right)^{\alpha+\beta} \left(\frac{v_0}{p}\right) \left(\frac{w}{p}\right) \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-1}{2} \cdot \frac{p-1}{2}} \left(\frac{p}{v_0}\right) \cdot (-1)^{\frac{w-1}{2} \cdot \frac{p-1}{2}} \left(\frac{p}{w}\right) \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{u^2+11v^2}{v_0}\right) \left(\frac{u^2+11v^2}{w}\right) \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{u^2}{v_0}\right) \left(\frac{u^2-121v^2+132v^2}{w}\right) \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{33}{w}\right) = \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{u-11v}{33}\right) \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{u+v}{3}\right) \left(\frac{u}{11}\right).
\end{aligned}$$

Thus, by Theorem 4.4 and the above we get

$$\begin{aligned}
P_{[\frac{p}{3}]} \left(\frac{u}{4v}\right) &\equiv -\left(\frac{p}{3}\right) \left(\frac{-11+u/v}{p}\right) \left(\frac{u}{11}\right) u \equiv -\left(\frac{u^2+11v^2}{3}\right) \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{u+v}{3}\right) u \\
&= \left(\frac{2}{p}\right)^{\alpha+\beta} (-1)^{\frac{v_0-w}{2} \cdot \frac{p-1}{2}} \left(\frac{v-u}{3}\right) u \pmod{p}.
\end{aligned}$$

This proves the corollary.

Let $p > 3$ be a prime and let \mathbb{F}_p be the field of p elements. For $m, n \in \mathbb{F}_p$ let $\#E_p(x^3 + mx + n)$ be the number of points on the curve $E: y^2 = x^3 + mx + n$ over the field \mathbb{F}_p . It is well known that (see for example [S1, pp.221-222])

$$(4.2) \quad \#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right).$$

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and the curve $y^2 = x^3 + mx + n$ has complex multiplication by K . By Deuring's theorem ([C, Theorem 14.16], [PV], [I]), we have

$$(4.3) \quad \#E_p(x^3 + mx + n) = \begin{cases} p + 1 & \text{if } p \text{ is inert in } K, \\ p + 1 - \pi - \bar{\pi} & \text{if } p = \pi\bar{\pi} \text{ in } K, \end{cases}$$

where π is in an order in K and $\bar{\pi}$ is the conjugate number of π . If $4p = u^2 + dv^2$ with $u, v \in \mathbb{Z}$, we may take $\pi = \frac{1}{2}(u + v\sqrt{-d})$. Thus,

$$(4.4) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) = \begin{cases} \pm u & \text{if } 4p = u^2 + dv^2 \text{ with } u, v \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In [Gr], [JM] and [PV] the sign of u in (4.4) was determined for those imaginary quadratic fields K with class number 1. In [LM] and [I] the sign of u in (4.4) was determined for imaginary quadratic fields K with class number 2. For general results on the sign of u in (4.4), see [M], [St], [RS] and the survey [Ri].

Lemma 4.6. *Let p be a prime with $p \equiv \pm 1 \pmod{8}$. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 6\sqrt{2})n + 24 - 14\sqrt{2}}{p} \right) \\ &= \begin{cases} 2x(\frac{2x}{3}) & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

Proof. It is easy to check the result for $p = 7$. Now assume $p \geq 17$. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-21 + 12\sqrt{2})x - 28 + 22\sqrt{2}$ has complex multiplication by the order of discriminant -24 . Since $4p = u^2 + 24v^2$ implies $2 \mid u$ and $p = (\frac{u}{2})^2 + 6v^2$, by (4.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &= \begin{cases} 2x(\frac{2x}{3})(\frac{1+\sqrt{2}}{p}) & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

Observe that

$$\frac{-15 - 6\sqrt{2}}{-21 + 12\sqrt{2}} = (1 + \sqrt{2})^2 \quad \text{and} \quad \frac{24 + 14\sqrt{2}}{-28 + 22\sqrt{2}} = (1 + \sqrt{2})^3.$$

Using Corollary 3.2 and Lemma 4.1 we see that

$$\begin{aligned} & \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 6\sqrt{2})n + 24 - 14\sqrt{2}}{p} \right) \\ &= \sum_{n=0}^{p-1} \left(\frac{n^3 - (15 + 6\sqrt{2})n + 24 + 14\sqrt{2}}{p} \right) \\ &= \left(\frac{1 + \sqrt{2}}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right). \end{aligned}$$

Now putting all the above together we obtain the result.

Theorem 4.5. *Let p be a prime such that $p \equiv 1, 7 \pmod{8}$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{\sqrt{2}}{2} \right) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. From Theorem 3.1 and Lemma 4.6 we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]} \left(\frac{\sqrt{2}}{2} \right) &\equiv - \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 6\sqrt{2})n + 24 - 14\sqrt{2}}{p} \right) \\ &\equiv \begin{cases} -2x(\frac{2x}{3}) = 2x(\frac{x}{3}) \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

Taking $m = 216$ and $t = \frac{\sqrt{2}}{2}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Remark 4.3 For any prime $p > 3$, Z.W. Sun conjectured that ([Su1, A14])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Conjecture 4.1. Let p be a prime such that $p \equiv 5, 11, 13, 19 \pmod{24}$. Then

$$P_{[\frac{p}{3}]} \left(\frac{\sqrt{2}}{2} \right) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 13, 19 \pmod{24}, \\ -2x(\frac{x}{3})\sqrt{2} \pmod{p} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}. \end{cases}$$

Lemma 4.7. Let p be a prime with $p \equiv \pm 1 \pmod{5}$. Then

$$\begin{aligned} &\sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 12\sqrt{5})n + 42 - 28\sqrt{5}}{p} \right) \\ &= \begin{cases} 2x(\frac{2x}{3}) & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Proof. From [I, Proposition 3.3] we know that the elliptic curve defined by the equation $y^2 = x^3 + (105 + 48\sqrt{5})x - 784 - 350\sqrt{5}$ has complex multiplication by the order of discriminant -15 . Since $4p = u^2 + 60v^2$ implies $2 \mid u$ and $p = (\frac{u}{2})^2 + 15v^2$, by (4.4) and [I, Theorem 3.1] we have

$$\begin{aligned} &\sum_{n=0}^{p-1} \left(\frac{n^3 + (105 + 48\sqrt{5})n - 784 - 350\sqrt{5}}{p} \right) \\ &= \begin{cases} 2x(\frac{2x}{3}) \left(\frac{(1+\sqrt{5})/2}{p} \right) & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Observe that

$$\frac{-15 + 12\sqrt{5}}{105 + 48\sqrt{5}} = (\sqrt{5} - 2)^2 \quad \text{and} \quad \frac{42 - 28\sqrt{5}}{-784 - 350\sqrt{5}} = (\sqrt{5} - 2)^3.$$

Using Lemma 4.1 we see that

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 12\sqrt{5})n + 42 - 28\sqrt{5}}{p} \right) \\ &= \left(\frac{\sqrt{5} - 2}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (105 + 48\sqrt{5})n - 784 - 350\sqrt{5}}{p} \right). \end{aligned}$$

Note that $(\sqrt{5} - 2) \frac{1+\sqrt{5}}{2} = (\frac{1-\sqrt{5}}{2})^2$. We then have $(\frac{\sqrt{5}-2}{p}) = (\frac{(1+\sqrt{5})/2}{p})$. Now putting all the above together we obtain the result.

Theorem 4.6. *Let p be a prime such that $p \equiv 1, 4 \pmod{5}$. Then*

$$P_{[\frac{p}{3}]}(\sqrt{5}) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod{p} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Proof. From Theorem 3.1 and Lemma 4.7 we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]}(\sqrt{5}) &\equiv -\left(\frac{p}{3}\right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-15 + 12\sqrt{5})n + 42 - 28\sqrt{5}}{p} \right) \\ &\equiv \begin{cases} -2x(\frac{2x}{3}) = 2x(\frac{x}{3}) \pmod{p} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Taking $m = -27$ and $t = \sqrt{5}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Conjecture 4.2. *Let p be an odd prime such that $p \equiv 2, 7, 8, 13 \pmod{15}$. Then*

$$P_{[\frac{p}{3}]}(\sqrt{5}) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 7, 13 \pmod{15}, \\ 2x(\frac{x}{3})\sqrt{5} \pmod{p} & \text{if } p = 5x^2 + 3y^2 \equiv 2, 8 \pmod{15}. \end{cases}$$

Lemma 4.8. *Let p be a prime such that $p \equiv \pm 1 \pmod{5}$. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-300 + 108\sqrt{5})n - 2520 + 1042\sqrt{5}}{p} \right) \\ &= \begin{cases} (\frac{2\sqrt{5}}{p})(\frac{x}{3})x & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Proof. From [I, p.134] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-2160 + 408\sqrt{5})x + 42130 - 10472\sqrt{5}$ has complex multiplication by the order of discriminant -75 . By (4.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-2160 + 408\sqrt{5})n + 42130 - 10472\sqrt{5}}{p} \right) \\ &= \begin{cases} \left(\frac{-25-13\sqrt{5}}{p} \right) \left(\frac{x}{3} \right) & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Observe that

$$\frac{-2160 + 408\sqrt{5}}{-300 + 108\sqrt{5}} = \left(-\frac{7 + \sqrt{5}}{2} \right)^2 \quad \text{and} \quad \frac{42130 - 10472\sqrt{5}}{-2520 + 1042\sqrt{5}} = \left(-\frac{7 + \sqrt{5}}{2} \right)^3.$$

Using Lemma 4.1 we see that

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-2160 + 408\sqrt{5})n + 42130 - 10472\sqrt{5}}{p} \right) \\ &= \left(\frac{-(7 + \sqrt{5})/2}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-300 + 108\sqrt{5})n - 2520 + 1042\sqrt{5}}{p} \right). \end{aligned}$$

Since

$$2(-7 - \sqrt{5})(-25 - 13\sqrt{5}) = 120 + 58\sqrt{5} = 2\sqrt{5}(29 + 12\sqrt{5}) = 2\sqrt{5}(3 + 2\sqrt{5})^2,$$

from the above we deduce the result.

Theorem 4.7. *Let p be a prime such that $p \equiv 1, 4 \pmod{5}$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{9}{20}\sqrt{5} \right) \equiv \begin{cases} -\left(\frac{x}{3} \right) \pmod{p} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \equiv \begin{cases} x^2 \pmod{p} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Proof. From Theorem 3.1 and Lemma 4.8 we deduce that

$$\begin{aligned} P_{[\frac{p}{3}]} \left(\frac{9}{20}\sqrt{5} \right) &\equiv -\left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + \frac{27-15\sqrt{5}}{\sqrt{5}}n + \frac{521-252\sqrt{5}}{20}}{p} \right) \\ &= -\left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{\left(\frac{n}{2\sqrt{5}} \right)^3 + \frac{27-15\sqrt{5}}{\sqrt{5}} \cdot \frac{n}{2\sqrt{5}} + \frac{521-252\sqrt{5}}{20}}{p} \right) \\ &= -\left(\frac{p}{3} \right) \left(\frac{2\sqrt{5}}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-300 + 108\sqrt{5})n - 2520 + 1042\sqrt{5}}{p} \right) \\ &\equiv \begin{cases} -\left(\frac{x}{3} \right) \pmod{p} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases} \end{aligned}$$

Taking $m = -8640$ and $t = \frac{9}{4\sqrt{5}}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Remark 4.4 In [S2] the author conjectured that for any prime $p > 5$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2 \text{ and } 5 \mid xy, \\ p - 2x^2 \pm 6xy \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2, 5 \nmid xy \\ & \text{and } x \equiv \pm y, \pm 2y \pmod{5}, \\ 0 \pmod{p^2} & \text{if } 3 \nmid p-1. \end{cases}$$

This is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } 4p = x^2 + 75y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } p \equiv 7, 13 \pmod{15} \text{ and so } 4p = 3x^2 + 25y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture 4.3. For any prime $p > 5$ we have

$$\sum_{k=0}^{p-1} \frac{9k+1}{(-8640)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv p \left(\frac{p}{15} \right) \pmod{p^3}.$$

Conjecture 4.4. Let p be a prime such that $p \equiv 7, 13, 17, 23 \pmod{30}$. Then

$$P_{[\frac{p}{3}]} \left(\frac{9}{20} \sqrt{5} \right) \equiv \begin{cases} 5y \left(\frac{y}{3} \right) \pmod{p} & \text{if } p \equiv 7, 13 \pmod{30} \text{ and so } 4p = 3x^2 + 25y^2, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30}. \end{cases}$$

Let $b \in \{17, 41, 89\}$ and $f(b) = -12^3, -48^3, -300^3$ according as $b = 17, 41, 89$. In [Su1, Conjectures A20, A22 and A23], Z.W. Sun conjectured that for any odd prime $p \neq 3, b$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{f(b)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3} \right) = \left(\frac{p}{b} \right) = 1 \text{ and so } 4p = x^2 + 3by^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3} \right) = \left(\frac{p}{b} \right) = -1 \text{ and so } 4p = 3x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{3} \right) = -\left(\frac{p}{b} \right). \end{cases}$$

Now we partially solve (4.5).

Theorem 4.8. Let p be an odd prime such that $\left(\frac{17}{p} \right) = 1$. Then

$$P_{[\frac{p}{3}]} \left(\frac{\sqrt{17}}{4} \right) \equiv \begin{cases} -\left(\frac{x}{3} \right) x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-12)^{3k}} \equiv \begin{cases} x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. From [I, p.134] we know that the elliptic curve defined by the equation $y^2 = x^3 - (60 + 12\sqrt{17})x - 210 - 56\sqrt{17}$ has complex multiplication by the order of discriminant -51 . Thus, by (4.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 - (60 + 12\sqrt{17})n - 210 - 56\sqrt{17}}{p} \right) \\ &= \begin{cases} \left(\frac{-2}{p} \right) \left(\frac{x}{3} \right) x & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

It then follows from (1.3) and Theorem 3.1 that

$$\begin{aligned} P_{[\frac{p}{3}]} \left(\frac{\sqrt{17}}{4} \right) &= \left(\frac{p}{3} \right) P_{[\frac{p}{3}]} \left(-\frac{\sqrt{17}}{4} \right) \equiv - \sum_{n=0}^{p-1} \left(\frac{n^3 + 3(-\sqrt{17} - 5)n + \frac{17}{4} + 22 + 7\sqrt{17}}{p} \right) \\ &= - \sum_{n=0}^{p-1} \left(\frac{(-\frac{n}{2})^3 - 3(5 + \sqrt{17})(-\frac{n}{2}) + \frac{105 + 28\sqrt{17}}{4}}{p} \right) \\ &= - \left(\frac{-2}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 - (60 + 12\sqrt{17})n - 210 - 56\sqrt{17}}{p} \right) \\ &\equiv \begin{cases} -\left(\frac{x}{3} \right) x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Taking $m = -12^3$ and $t = \frac{\sqrt{17}}{4}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Conjecture 4.5. *Let p be an odd prime such that $(\frac{p}{17}) = -1$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{\sqrt{17}}{4} \right) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ -\left(\frac{y}{3} \right) y \sqrt{17} \pmod{p} & \text{if } p \equiv 2 \pmod{3} \text{ and so } 4p = 3x^2 + 17y^2. \end{cases}$$

Theorem 4.9. *Let p be an odd prime such that $(\frac{41}{p}) = 1$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{5\sqrt{41}}{32} \right) \equiv \begin{cases} -\left(\frac{x}{3} \right) x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 123y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-48)^{3k}} \equiv \begin{cases} x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 123y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. From [I, p.134] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-960 + 120\sqrt{41})x - 13314 + 2240\sqrt{41}$ has complex multiplication by the order of discriminant -123 . Thus, by [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 - (960 - 120\sqrt{41})n - 13314 + 2240\sqrt{41}}{p} \right) \\ &= \begin{cases} \left(\frac{-2}{p} \right) \left(\frac{x}{3} \right) x & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 123y^2, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

It then follows from (1.3) and Theorem 3.1 that

$$\begin{aligned}
P_{[\frac{p}{3}]} \left(\frac{5\sqrt{41}}{32} \right) &\equiv - \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + 3(\frac{5}{8}\sqrt{41} - 5)n + (\frac{5\sqrt{41}}{16})^2 - 7 \cdot \frac{5\sqrt{41}}{8} + 22}{p} \right) \\
&= - \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{(-\frac{n}{8})^3 + \frac{15}{8}(\sqrt{41} - 8)(-\frac{n}{8}) + \frac{6657 - 1120\sqrt{41}}{256}}{p} \right) \\
&= - \left(\frac{p}{3} \right) \left(\frac{-2}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (960 - 120\sqrt{41})n - 13314 + 2240\sqrt{41}}{p} \right) \\
&\equiv \begin{cases} -(\frac{x}{3})x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Taking $m = -48^3$ and $t = \frac{5\sqrt{41}}{32}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Conjecture 4.6. *Let p be an odd prime such that $(\frac{p}{41}) = -1$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{5\sqrt{41}}{32} \right) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ -(\frac{y}{3})y\sqrt{41} \pmod{p} & \text{if } p \equiv 2 \pmod{3} \text{ and so } 4p = 3x^2 + 41y^2. \end{cases}$$

Theorem 4.10. *Let $p > 5$ be a prime such that $(\frac{89}{p}) = 1$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{53\sqrt{89}}{500} \right) \equiv \begin{cases} -(\frac{x}{3})x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 267y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-300)^{3k}} \equiv \begin{cases} x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 267y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. From [I, p.135] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-37500 + 3180\sqrt{89})x + 3250002 - 371000\sqrt{89}$ has complex multiplication by the order of discriminant -267 . Thus, by [I, Theorem 3.1] we have

$$\begin{aligned}
&\sum_{n=0}^{p-1} \left(\frac{n^3 + (-37500 + 3180\sqrt{89})n + 3250002 - 371000\sqrt{89}}{p} \right) \\
&= \begin{cases} (\frac{2}{p})(\frac{x}{3})x & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 267y^2, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

It then follows from (1.3) and Theorem 3.1 that

$$\begin{aligned}
P_{[\frac{p}{3}]} \left(\frac{53\sqrt{89}}{500} \right) &\equiv - \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + 3(4 \cdot \frac{53}{500}\sqrt{89} - 5)n + 4(\frac{53}{500}\sqrt{89})^2 - 28 \cdot \frac{53\sqrt{89}}{500} + 22}{p} \right) \\
&= - \left(\frac{p}{3} \right) \sum_{n=0}^{p-1} \left(\frac{(\frac{n}{50})^3 + \frac{-1875+159\sqrt{89}}{125} \cdot \frac{n}{50} + \frac{1625001-185500\sqrt{89}}{250^2}}{p} \right) \\
&= - \left(\frac{p}{3} \right) \left(\frac{50}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-37500 + 3180\sqrt{89})n + 3250002 - 371000\sqrt{89}}{p} \right) \\
&\equiv \begin{cases} -(\frac{x}{3})x \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = x^2 + 51y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Taking $m = -300^3$ and $t = \frac{53\sqrt{89}}{500}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Conjecture 4.7. *Let p be an odd prime such that $(\frac{p}{89}) = -1$. Then*

$$P_{[\frac{p}{3}]} \left(\frac{53\sqrt{89}}{500} \right) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ -(\frac{y}{3})y\sqrt{89} \pmod{p} & \text{if } p \equiv 2 \pmod{3} \text{ and so } 4p = 3x^2 + 89y^2. \end{cases}$$

In the end we pose the following conjectures.

Conjecture 4.8. *Let p be a prime with $p \equiv \pm 1 \pmod{5}$. Then*

$$\begin{aligned}
&\sum_{n=0}^{p-1} \left(\frac{n^3 + 3(-125 + 44\sqrt{5})n + 154(21 - 10\sqrt{5})}{p} \right) \\
&= \begin{cases} 2x(\frac{2x}{3}) & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}
\end{aligned}$$

If Conjecture 4.8 is true, we may discuss the following conjecture in [S2]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{15^{3k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases}$$

Conjecture 4.9. *For any prime $p > 5$ we have*

$$\sum_{k=0}^{p-1} \frac{33k+4}{15^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4p \left(\frac{p}{3} \right) \pmod{p^3}.$$

Conjecture 4.10. *Let $p > 5$ be a prime. Then*

$$P_{[\frac{p}{3}]} \left(\frac{11\sqrt{5}}{25} \right) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod{p} & \text{if } p \equiv 1, 19 \pmod{30} \text{ and so } p = x^2 + 15y^2, \\ -2y(\frac{y}{3})\sqrt{5} \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30} \text{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod{p} & \text{if } p \equiv 7, 11, 13, 29 \pmod{30}. \end{cases}$$

In [S2], the author also conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture 4.11. *For any prime $p > 3$ we have*

$$\sum_{k=0}^{p-1} \frac{15k+2}{1458^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv (-1)^{\frac{p-1}{2}} 2p \pmod{p^3}.$$

Conjecture 4.12. *Let $p > 3$ be a prime. Then*

$$P_{[\frac{p}{3}]} \left(\frac{5\sqrt{3}}{9} \right) \equiv \begin{cases} (-1)^{\frac{p-1}{2}} 2x(\frac{x}{3}) \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

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